ISRAEL JOURNAL OF MATHEMATICS 84 (1993), 119–128

CARATHÉODORY BALLS AND NORM BALLS OF THE DOMAIN $H = \{(z_1, z_2) \in C^2: |z_1| + |z_2| < 1\}$

BY

BINYAMIN SCHWARZ

Department of Mathematics
Technion—Israel Institute of Technology, Haifa 32000, Israel

ABSTRACT

Let H be the domain in C^2 defined by

$$H = \{\mathbf{z} = (z_1, z_2): \|\mathbf{z}\|_1 = |z_1| + |z_2| < 1\}.$$

Let $C_H(\mathbf{z}, \mathbf{w})$ be the Carathéodory distance of H, $\mathbf{z}, \mathbf{w} \in H$. The Carathéodory ball $B_C(\mathbf{z}_C, \alpha; H)$ with center $\mathbf{z}_C$, $\mathbf{z}_C \in H$, and radius α, $0 < \alpha < 1$, is defined by $B_C(\mathbf{z}_C, \alpha; H) = \{\mathbf{z}: C_H(\mathbf{z}, \mathbf{z}_C) < \operatorname{arc\,tanh} \alpha\}$. The norm ball $B_N(\mathbf{z}_N, r)$ with center $\mathbf{z}_N$, $\mathbf{z}_N \in H$, and radius r, $0 < r < 1 - \|\mathbf{z}_N\|_1$, is defined by $B_N(\mathbf{z}_N, r) = \{\mathbf{z}: \|\mathbf{z} - \mathbf{z}_N\|_1 < r\}$.

THEOREM: *The only Carathéodory balls of H which are also norm balls are those with their center at the origin.*

1. Introduction

We start this introduction with the definition of the Carathéodory distance for bounded domains in C^n and define Carathéodory balls. For any given norm in C^n we define the norm balls. We quote three theorems related to these notions. We close the introduction with three examples of unit balls for which it is known which Carathéodory balls are also norm balls. In Section 2 we state four lemmas. The first three are not new, and the last lemma is purely geometric. We use these lemmas in the last section to prove the following result. Let $H \subset C^2$ be the unit ball with respect to the l_1 norm. The only Carathéodory balls of H which are also norm balls are those with their center at the origin.

Received October 29, 1992

We denote the Poincaré (non-Euclidean) distance of the unit disc

$$\Delta = \{z\colon |z| < 1\}$$

of C^1 by $\rho(z,w)$:

$$\rho(z,w) = \operatorname{arc\,tanh} \left\| \frac{z-w}{1-z\bar{w}} \right\|, \qquad |z|<1, \quad |w|<1. \tag{1.1}$$

Let D be a bounded domain in the n-dimensional complex space C^n and let $\mathrm{Hol}(D,\Delta)$ be the set of all holomorphic functions $f(\mathbf{z}) = f(z_1,\ldots,z_n)$ from D into Δ. The **Carathéodory distance** $C_D(\mathbf{z},\mathbf{w})$ of D, $\mathbf{z},\mathbf{w} \in D$, is defined by

$$C_D(\mathbf{z},\mathbf{w}) = \sup \rho(f(\mathbf{z}), f(\mathbf{w})), \tag{1.2}$$

where the supremum is taken over $\mathrm{Hol}(D,\Delta)$. The Schwarz–Pick theorem implies that

$$C_\Delta(z,w) = \rho(z,w). \tag{1.3}$$

The Carathéodory distance is thus a generalization of the Poincaré distance of the unit disc to general multidimensional domains. For a given bounded domain D, $D \subset C^n$, and any $\mathbf{z}_c \in D$, and any α, $0<\alpha<1$, we set

$$B_C = B_C(\mathbf{z}_c,\alpha;D) = \{\mathbf{z}\colon C_D(\mathbf{z},\mathbf{z}_c) < \operatorname{arc\,tanh}\alpha\}, \tag{1.4}$$

and call B_C **the Carathéodory ball in D with center $\mathbf{z}_c$ and radius α.**

We shall use, implicitly, the following fact, cf. [D, p.88].

THEOREM 1.1: *Let D be a convex bounded domain in C^n. Then for any Carathéodory ball $B_C = B_C(\mathbf{z}_c,\alpha;D)$, $\mathbf{z}_c \in D$, $0<\alpha<1$, its closure $\bar{B}_C \subset D$.*

Next, we quote a special case of a result of Vesentini [V, Lemma 3.5; D, Proposition 6.20].

THEOREM 1.2: *Let $\|\cdot\|$ be a given vector norm in C^n and let B be the corresponding unit ball*

$$B = \{\mathbf{z}\colon \|\mathbf{z}\| < 1\}. \tag{1.5}$$

Then, for any $\mathbf{z} \in B$ and any $a \in C^1$ such that $a\mathbf{z} \in B$,

$$C_B(\mathbf{z},a\mathbf{z}) = \rho(\|\mathbf{z}\|, a\|\mathbf{z}\|). \tag{1.6}$$

Let $\|\cdot\|$ be a given norm in C^n. For any $\mathbf{z}_N \in C^n$ and any positive r we set

$$B_N = B_N(\mathbf{z}_N, r) = \{\mathbf{z}\colon \|\mathbf{z} - \mathbf{z}_N\| < r\}, \tag{1.7}$$

and call B_N **the norm ball with center $\mathbf{z}_N$ and radius** r. By (1.5) $B = B_N(O, 1)$ where O is the origin of C^n.

THEOREM 1.3: *Let $\|\cdot\|$ be a given norm in C^n and let B be the corresponding unit ball. Then, for any α, $0 < \alpha < 1$,*

$$B_C(O, \alpha; B) = B_N(O, \alpha). \tag{1.8}$$

This follows from (1.6) with $a = 0$, and (1.1). For a direct proof see [FV, Theorem IV.1.8].

In the one-dimensional case every norm disc is an ordinary (Euclidean) disc and in this case every non-Euclidean (Poincaré) disc of Δ is a Euclidean disc (cf. Lemma 2.1 below). For higher dimensions, Theorem 1.3 shows that both kinds of balls coincide if their center is at the origin. The question arises for which norms also other balls are of both kinds. The following results are known [S1, Section 5]: (i) For the Euclidean unit ball $B^n = \{\mathbf{z}\colon \|\mathbf{z}\|_2 < 1\}$ of C^n, every Carathéodory ball with center $\mathbf{z}_c \neq O$ is an ellipsoid and not an ordinary, Euclidean, ball. (ii) For the polydisc $P^n = \{\mathbf{z}\colon \|\mathbf{z}\|_\infty < 1\}$ of C^n, a Carathéodory ball with center $\mathbf{z}_c = (z_1^c, \dots, z_n^c)$ is a l_∞ norm ball if and only if $|z_1^c| = \cdots = |z_n^c|$ (cf. Eq. (2.4) below). Finally, (iii) for the set of all $n \times n$ complex matrices P, considered as lying in C^{n^2}, with spectral norm $\|P\| < 1$, a set, depending on $n^2 + 2$ real parameters, of Carathéodory balls are also balls with respect to the spectral norm. Note that in these three cases the unit balls B are homogeneous domains, i.e., the group of automorphisms is transitive, and known. So by using an automorphism φ such that $\varphi(O) = \mathbf{z}_c$, we obtain that $\varphi(B_C(O, \alpha; B)) = B_C(\mathbf{z}_c, \alpha; B)$. As $B_C(O, \alpha; B)$ is, by (1.8), known, we have to examine when its image is a norm ball. In the case which we are about to consider, i.e., for the two-dimensional unit ball of the l_1 norm

$$H = \{\mathbf{z} = (z_1, z_2)\colon \|\mathbf{z}\|_1 = |z_1| + |z_2| < 1\}, \tag{1.9}$$

the group of automorphisms is highly intransitive. Indeed, it was already shown in [K] (cf. also [T]) that every automorphism of H keeps the origin fixed. We have thus to prove our result in a different way and in the next section we bring four lemmas needed for the proof. We remark that in [HP2] this domain is denoted by D_{22}. Thullen [T] called it, for an obvious reason, Hyperkegel; hence our notation.

2. Four lemmas

LEMMA 2.1: *Let the Poincaré (non-Euclidean) circle $\Gamma_p = \Gamma(z_p, \alpha)$, $\Gamma_p \subset \Delta$, be defined by*

$$\Gamma(z_p,\alpha) = \left\{z\colon \left\|\frac{z - z_p}{1 - z\bar{z}_p}\right\| = \alpha\right\}, \qquad z_p \in \Delta,\ 0 < \alpha < 1. \tag{2.1}$$

Γ_p is also a Euclidean circle $\gamma_E = \gamma(z_E, r)$,

$$\gamma(z_E, r) = \{z\colon |z - z_E| < r\}, \tag{2.2}$$

where

$$z_E = z_p \frac{1-\alpha^2}{1-\alpha^2|z_p|^2}, \tag{2.3}$$

and

$$r = \alpha\frac{1-|z_p|^2}{1-\alpha^2|z_p|^2}. \tag{2.4}$$

For the proof assume first that $z_p = x_p$ is positive and determine the real points x_1 and x_2 on $\Gamma(x_p, \alpha)$, $-1 < x_1 < x_p < x_2 < 1$. Then $z_E = (x_1 + x_2)/2$, $r = (x_2 - x_1)/2$, and the general result follows by rotation.

LEMMA 2.2: *Let $\tilde{\mathbf{z}} \in H$, $a \in C^1$, and $a\tilde{\mathbf{z}} \in H$. Then*

$$C_H(\tilde{\mathbf{z}}, a\tilde{\mathbf{z}}) = \rho(\|\tilde{\mathbf{z}}\|_1, a\|\tilde{\mathbf{z}}\|_1). \tag{2.5}$$

This is a special case of Theorem 1.2. For a direct proof set

$$g(u) = f((u/\|\tilde{\mathbf{z}}\|_1)\tilde{\mathbf{z}}).$$

For $f \in \mathrm{Hol}(H, \Delta)$, $g \in \mathrm{Hol}(\Delta, \Delta)$, (1.2) and (1.3) imply

$$C_H(\tilde{\mathbf{z}}, a\tilde{\mathbf{z}}) \le \rho(\|\tilde{\mathbf{z}}\|_1, a\|\tilde{\mathbf{z}}\|_1).$$

The function $f_0(\mathbf{z}) = z_1 e^{-i\theta_1} + z_2 e^{-i\theta_2}$, where $\tilde{\mathbf{z}} = (\tilde{z}_1, \tilde{z}_2) = (|\tilde{z}_1|e^{i\theta_1}, |\tilde{z}_1|e^{i\theta_2})$, yields now the equality sign in (2.5).

LEMMA 2.3: *Let the points* $\mathbf{z} = (z_1, z_2)$ *and* $\tilde{\mathbf{z}} = (w_1, z_2)$ *be in* H. *Then*

$$C_H(\mathbf{z}, \tilde{\mathbf{z}}) = \rho\Big(\frac{z_1}{1-|z_2|}, \frac{w_1}{1-|z_2|}\Big). \tag{2.6}$$

This follows from a result of Gentili on "linear" complex geodesics [G; p.45]; for an elementary proof see [S2, Lemma 1]. We remark that both proofs yield a similar result for $C_{H^n}(\mathbf{z}, \tilde{\mathbf{z}})$, where H^n is the unit ball of the l_1 norm in C^n and $\mathbf{z}$ and $\tilde{\mathbf{z}}$ are points in H^n differing only in one coordinate.

LEMMA 2.4: *Let* γ_1 *and* γ_2 *be two (Euclidean) circles in the complex plane* C^1:

$$\begin{aligned} \gamma_1 &= \big\{z \colon z = z_1(\varphi) = z_1 + r_1 e^{i(\varphi+\theta_1)},\ r_1 > 0,\ -\infty < \varphi < \infty\big\}, \\ \gamma_2 &= \big\{z \colon z = z_2(\varphi) = z_2 + r_2 e^{i(\varphi+\theta_2)},\ r_2 > 0,\ -\infty < \varphi < \infty\big\}. \end{aligned} \tag{2.7}$$

A necessary and sufficient condition for the existence of two points ζ_1 *and* ζ_2 *in* C^1 *and two real constants* θ_1 *and* θ_2 *such that the equality*

$$|z_1(\varphi) - \zeta_1| + |z_2(\varphi) - \zeta_2| = \text{const}, \qquad -\infty < \varphi < \infty, \tag{2.8}$$

holds is

$$\zeta_1 = z_1, \quad \zeta_2 = z_2. \tag{2.9}$$

Proof: Sufficiency of (2.9) is obvious. To prove necessity, we remark that if $\zeta_1 = z_1$ and (2.8) holds, then clearly also $\zeta_2 = z_2$. We thus assume, by negation,

$$\zeta_1 \neq z_1, \quad \zeta_2 \neq z_2, \tag{2.10}$$

and denote

$$|\zeta_k - z_k| = d_k, \quad k = 1, 2. \tag{2.11}$$

So (2.10) becomes

$$d_1 > 0, \quad d_2 > 0. \tag{2.10'}$$

Let N_k be the point on γ_k nearest to ζ_k, and let F_k be the point on γ_k farthest away from ζ_k, $k = 1, 2$. If (2.8) holds, then we use the notation $z_1(\varphi) \sim z_2(\varphi)$, $-\infty < \varphi < \infty$, for pairs of corresponding points. It follows that

$$N_1 \sim F_2, \quad N_2 \sim F_1. \tag{2.12}$$

Indeed, near N_1, the distance $|z_1(\varphi) - \zeta_1|$ increases both for decreasing and increasing values of φ. For any point $z_2(\varphi^*) \neq F_2$, the distance $|z_2(\varphi) - \zeta_2|$ increases, as φ moves near φ^* in, at least, one direction. So if N_1 would not correspond to F_2, (2.8) could not be true. After an appropriate choice of θ_1 and θ_2 we thus have

$$(2.13) \qquad N_1 = z_1(\varphi_1), \quad F_1 = z_1(\varphi_1 + \pi), \quad N_2 = z_2(\varphi_1 + \pi), \quad F_2 = z_2(\varphi_1)$$

(φ is always taken mod 2π).

We have to distinguish between the cases (a) $0 < d_k \leq r_k$ and (b) $r_k < d_k$ for each k, $k = 1, 2$. In case (a) $|N_k - \zeta_k| = r_k - d_k$, $|F_k - \zeta_k| = r_k + d_k$. In case (b) $|N_k - \zeta_k| = d_k - r_k$, $|F_k - \zeta_k| = r_k + d_k$. We also consider the points Q_k, halfway between F_k and N_k: $Q_k = z_k(\varphi_1 + \pi/2)$, $k = 1, 2$. In both cases $|Q_k - \zeta_k| = (d_k^2 + r_k^2)^{1/2}$.

There are now three possibilities:

(i) $d_1 \leq r_1$, $d_2 \leq r_2$. Then $|N_1 - \zeta_1| + |F_2 - \zeta_2| = r_1 - d_1 + r_2 + d_2$, and $|F_1 - \zeta_1| + |N_2 - \zeta_2| = r_1 + d_1 + r_2 - d_2$. So if (2.8) were true, we would obtain $d_1 = d_2$ and $|z_1(\varphi) - \zeta_1| + |z_2(\varphi) - \zeta_2| = r_1 + r_2$. However, $|Q_1 - \zeta_1| + |Q_2 - \zeta_2| = (d_1^2 + r_1^2)^{1/2} + (d_2^2 + r_2^2)^{1/2} > r_1 + r_2$.

(ii) $d_1 \leq r_1$, $d_2 > r_2$. Now $|N_1 - \zeta_1| + |F_2 - \zeta_2| = r_1 - d_1 + r_2 + d_2$, and $|F_1 - \zeta_1| + |N_2 - \zeta_2| = r_1 + d_1 + d_2 - r_2$. (2.8) would give $d_1 = r_2$ and $|z_1(\varphi) - \zeta_1| + |z_2(\varphi) - \zeta_2| = r_1 + d_2$, and this is again smaller than $(d_1^2 + r_1^2)^{1/2} + (d_2^2 + r_2^2)^{1/2}$.

(iii) $d_1 > r_1$, $d_2 > r_2$. Now we would obtain $d_1 - r_1 + r_2 + d_2 = d_1 + r_1 + d_2 - r_2$. Hence $r_1 = r_2$, and again $(d_1^2 + r_1^2)^{1/2} + (d_2^2 + r_2^2)^{1/2} > d_1 + d_2$. This completes the proof. ∎

We add a conjecture and a remark.

(i) The conjecture is that the lemma can be generalized to hold for any number n, $n \geq 2$, of circles γ_k and points ζ_k.

(ii) It is easily seen that the analogue of this lemma does not hold for the l_2 norm; $|z_1(\varphi) - \zeta_1|^2 + |z_2(\varphi) - \zeta_2|^2 = \text{const}$, $-\infty < \varphi < \infty$, holds, for appropriate choice of θ_1 and θ_2, whenever $r_1 d_1 = r_2 d_2$.

3. Carathéodory and norm balls of H

THEOREM 3.1: *Let $H \subset C^2$ be the unit ball with respect to the l_1 norm:*

$$H = \{\mathbf{z} = (z_1, z_2)\colon \|\mathbf{z}\|_1 = |z_1| + |z_2| < 1\}. \tag{1.9}$$

The only Carathéodory balls of H which are also norm balls are those with their center at the origin.

Proof: Let $B_C = B_C(\mathbf{z}_c, \alpha; H)$, $\mathbf{z}_c \in H$, $0 < \alpha < 1$, be a Carathéodory ball of H and assume that

$$\mathbf{z}_c \neq O. \tag{3.1}$$

We denote

$$\mathbf{z}_c = (z_1^c, z_2^c) \tag{3.2}$$

and assume first that

$$z_1^c \neq 0, \quad z_2^c \neq 0. \tag{3.1'}$$

All points $a\mathbf{z}_c$, $a \in C^1$, satisfying

$$C_H(\mathbf{z}_c, a\mathbf{z}_c) = \operatorname{arc\,tanh} \alpha, \tag{3.3}$$

lie on ∂B_C. By Lemma 2.2 this yields

$$\rho(\|\mathbf{z}_c\|_1, a\|\mathbf{z}_c\|_1) = \operatorname{arc\,tanh} \alpha. \tag{3.4}$$

$a\|\mathbf{z}_c\|_1$, with variable a and the given $\|\mathbf{z}_c\|_1$, lies thus on the non-Euclidean circle $\Gamma_p = \Gamma(\|\mathbf{z}_c\|_1, \alpha)$. By Lemma 2.1 this is also a Euclidean circle $\gamma_E = \gamma(\hat{z}_E, \hat{r})$, where

$$\hat{z}_E = \|\mathbf{z}_c\|_1 \frac{1 - \alpha^2}{1 - \alpha^2 \|\mathbf{z}_c\|_1^2} \tag{3.5}$$

and

$$\hat{r} = \alpha \frac{1 - \|\mathbf{z}_c\|_1^2}{1 - \alpha^2 \|\mathbf{z}_c\|_1^2}. \tag{3.6}$$

It follows that az_k^c, $k=1,2$, describes a Euclidean circle $\gamma_k=\gamma(\hat{z}_k,r_k)$, where

$$\hat{z}_k = z_k^c \frac{1-\alpha^2}{1-\alpha^2\|\mathbf{z}_c\|_1^2}, \qquad k=1,2, \tag{3.7}$$

and

$$r_k = \frac{|z_k^c|}{\|\mathbf{z}_c\|_1}\hat{r}, \qquad k=1,2. \tag{3.8}$$

Note that $r_1+r_2=\hat{r}$, and note also that $a\|\mathbf{z}_c\|_1 = \hat{z}_E+\hat{r}e^{i\varphi}$ implies $az_k^c = \hat{z}_k + r_k e^{i(\varphi+\theta_k)}$, where $z_k^c = |z_k^c|e^{i\theta_k}$, $k=1,2$.

If $\partial B_C(\mathbf{z}_c,\alpha;H)$ is also the boundary ∂B_N of a norm ball $B_N(\mathbf{z}_N,r)$ of H, then, in particular, the points $a\mathbf{z}_c=(az_1^c,az_2^c)$ satisfying (3.3) have to lie on this boundary $\partial B_N(\mathbf{z}_N,r)$. By Lemma 2.4 this can happen only if

$$\mathbf{z}_N = \hat{\mathbf{z}} = (\hat{z}_1,\hat{z}_2), \tag{3.9}$$

where $\hat{z}_1$ and $\hat{z}_2$ are given by (3.7), and if $r=\hat{r}$, given by (3.6).

We now show that not all points of $\partial B_C(\mathbf{z}_c,\alpha;H)$ lie on $\partial B_N(\hat{\mathbf{z}},\hat{r})$. To do this, we choose the points

$$\tilde{\mathbf{z}} = (w_1, z_2^c), \tag{3.10}$$

with variable w_1, and z_2^c given by (3.2), for which

$$C_H(\mathbf{z}_c,\tilde{\mathbf{z}}) = \operatorname{arc\,tanh}\alpha. \tag{3.11}$$

Lemma 2.3 implies $\rho(z_1^c/(1-|z_2^c|),\, w_1/(1-|z_2^c|)) = \operatorname{arc\,tanh}\alpha$. $w_1/(1-|z_2^c|)$ lies thus on the non-Euclidean circle $\Gamma(z_1^c/(1-|z_2^c|),\alpha)$. This is a Euclidean circle with center at $\{z_1^c(1-|z_2^c|)(1-\alpha^2)\}/\{(1-|z_2^c|)^2-\alpha^2|z_1^c|^2\}$ and a given radius ρ. Hence w_1 lies on a Euclidean circle $\gamma_E=\gamma(z_E,r)$ where

$$z_E = \frac{z_1^c(1-|z_2^c|)^2(1-\alpha^2)}{(1-|z_2^c|)^2-\alpha^2|z_1^c|^2}, \tag{3.12}$$

and $r=(1-|z_2^c|)\rho$. If $\tilde{\mathbf{z}}$ lies on $\partial B_N(\hat{\mathbf{z}},\hat{r})$, then the l_1 distance $\|\tilde{\mathbf{z}}-\hat{\mathbf{z}}\|_1 = |w_1-\hat{z}_1|+|z_2^c-\hat{z}_2|$ has to be constant ($=\hat{r}$) for all w_1 on γ_E. This can only happen if the center z_E of γ_E equals $\hat{z}_1$. To show that this is impossible, assume, by negation, that

$$z_E = \hat{z}_1. \tag{3.13}$$

(3.7), (3.12), (3.13) and $z_1^c \neq 0$, $0 < \alpha < 1$ yield $|z_2^c| = |z_2^c|(|z_1^c| + |z_2^c|)$. As we also assumed that $z_2^c \neq 0$, we obtain $\|\mathbf{z}_c\|_1 = 1$ which contradicts the assumption $\mathbf{z}_c \in H$. This concludes the proof in case (3.1′) holds.

Assume now that

$$z_1^c = 0, \quad z_2^c \neq 0, \tag{3.1''}$$

holds; so from now on $\mathbf{z}_c$ is of the form

$$\mathbf{z}_c = (0, z_2^c), \quad z_2^c \neq 0. \tag{3.2'}$$

The points $a\mathbf{z}_c = (0, az_2^c)$, which satisfy (3.3), lie on $\partial B_C(\mathbf{z}_c, \alpha; H)$. So, if they also lie on the boundary ∂B_N of a norm ball, then, again, necessarily $B_N = B_N(\hat{\mathbf{z}}', \hat{r}')$, where

$$\hat{\mathbf{z}}' = (0, \hat{z}_2'), \tag{3.9'}$$

$$\hat{z}_2' = z_2^c \frac{1-\alpha^2}{1-\alpha^2|z_2^c|^2}, \tag{3.7'}$$

and

$$\hat{r}' = \alpha \frac{1-|z_2^c|^2}{1-\alpha^2|z_2^c|^2}. \tag{3.6'}$$

Consider again the points $\tilde{\mathbf{z}} = (w_1, z_2^c)$ which satisfy (3.11). $w_1/(1-|z_2^c|)$ lies on the circle $\Gamma_p = \Gamma(0,\alpha)$ which is also the Euclidean circle $\gamma_E = \gamma(0,\alpha)$. w_1 lies therefore on the Euclidean circle $\gamma(0,(1-|z_2^c|)\alpha)$. We thus obtain

$$\begin{aligned} \|\tilde{\mathbf{z}} - \hat{\mathbf{z}}'\|_1 &= |w_1| + |z_2^c - \hat{z}_2'| \\ &= (1-|z_2^c|)\alpha + |z_2^c|\{1-(1-\alpha^2)/(1-\alpha^2|z_2^c|^2)\} = r^*. \end{aligned} \tag{3.14}$$

(3.6′) and (3.14) yield

$$(\hat{r}' - r^*)(1-\alpha^2|z_2^c|^2)/(\alpha|z_2^c|) = (\alpha-\alpha^2)|z_2^c|^2 + (\alpha^2-1)|z_2^c| + (1-\alpha). \tag{3.15}$$

Let us denote the right hand side of (3.15) by $f_\alpha(|z_2^c|)$, $0 < \alpha < 1$. $f_\alpha(0) = 1-\alpha$, $f_\alpha(1) = 0$, and $f_\alpha'(|z_2^c|) < 0$ for $0 < |z_2^c| < 1$. Hence $f_\alpha(|z_2^c|) > 0$ for $0 < \alpha < 1$, $0 < |z_2^c| < 1$, and thus $\hat{r}' > r^*$. The points $\tilde{\mathbf{z}}$ do not lie on $\partial B_N(\hat{\mathbf{z}}', \hat{r}')$. This completes the proof of the theorem. ∎

We conclude with two remarks.

(i) Recently Hahn and Pflug [HP1] observed that the transformation

$$w_1 = z_1 + z_2, \quad w_2 = i(z_2 - z_1), \tag{3.16}$$

maps H onto the unit ball B_2^*, $B_2^* \subset C^2$, given by the norm

$$N^*(\mathbf{w}) = \frac{1}{\sqrt{2}}(|w_1|^2 + |w_2|^2 + |w_1^2 + w_2^2|)^{1/2}, \quad \mathbf{w} = (w_1, w_2).$$

The transformation (3.16) is biholomorphic and isometric, so the conclusion of Theorem 3.1 holds also for B_2^*.

(ii) We do not know if Theorem 3.1 can be generalized to hold for the l_1 unit ball of C^n, $n > 2$. If, as conjectured, Lemma 2.4 can be generalized, then the proof of the generalized theorem is virtually the same as the one brought here for the l_1 unit ball H of C^2.

References

[D] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.

[FV] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.

[G] G. Gentili, *Regular complex geodesics for the domain* $D_n = \{(z_1, \ldots, z_n) \in C^n : |z_1| + \cdots + |z_n| < 1\}$, Lecture Notes in Math. **1275**, Springer, Berlin, 1987, pp. 35-45.

[HP1] K.T. Hahn and P. Pflug, *On a minimal complex norm that extends the real Euclidean norm*, Monatsh. Math. **105** (1988), 107–112.

[HP2] K.T. Hahn and P. Pflug, *The Kobayashi and Bergman metrics on generalized Thullen domains*, Proc. Amer. Math. Soc. **104** (1988), 207–214.

[K] N. Kritikos, *Über analytische Abbildungen einer Klasse von vierdimensionalen Gebieten*, Math. Ann. **99** (1928), 321–341.

[S1] B. Schwarz, *On the geometry of some matrix domains*, Analysis **10** (1990), 319–335.

[S2] B. Schwarz, *Bounds for the Carathéodory distance of the unit ball*, Linear and Multilinear Algebra **32** (1992), 93–101.

[T] P. Thullen, *Zu den Abbildungen durch analytische Funktionen mehrerer komplexer Veränderlichen*, Math. Ann. **104** (1931), 244–259.

[V] E. Vesentini, *Complex geodesics*, Compositio Mathematica **44** (1981), 375–394.